Tutorial 6 (21 Oct)

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Q1) Prove the Hölder's and Minkowski's Inequalities for $\mathbb{R}^n$ in the following sense:

Fix a conjugate pair (p,q), i.e. $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$.

Define $\|\cdot\|_p : \mathbb{R}^n \to \mathbb{R}$ as $\|(x_1,\cdots,x_n)\|_p := \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}}$. Similarly for $\|\cdot\|_q$.

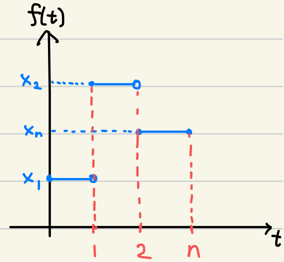
(a) Show that Hölder's Inequality holds: for any $x,y\in\mathbb{R}^n$, $\sum_{i=1}^n |x_i y_i| \leq \|x\|_p \|y\|_q$

(b) Show that Minkowski's Inequality holds: for any $x,y\in\mathbb{R}^n$, $\|x+y\|_p \leq \|x\|_p + \|y\|_p$

Sol) (a) Idea: Apply Hölder's Inequality for $R[0,n]$ to some step functions.

Recall Hölder's Inequality for $R[0,n]$: for any $f,g\in R[0,n]$, $\|fg\|_1 \leq \|f\|_p\|g\|_q$.

Given $x,y\in\mathbb{R}^n$, define f,g as
$$\begin{cases} f(t) := x_i \text{ for } t\in[i-1,i),\ i=1,\cdots,n \text{ and } f(n):=x_n. \\ g(t) := y_i \text{ for } t\in[i-1,i),\ i=1,\cdots,n \text{ and } g(n):=y_n. \end{cases}$$



then $(fg)(t) = x_i y_i$ for $t\in[i-1,i)$, $i=1,\cdots,n$ and $(fg)(n) = x_n y_n$.

Note that
- $\|fg\|_1 = \int_0^n |(fg)(t)|\,dt = \sum_{i=1}^n \int_{i-1}^i |x_i y_i|\,dt = \sum_{i=1}^n |x_i y_i|$.
- $\|f\|_p = \left(\int_0^n |f(t)|^p dt\right)^{\frac{1}{p}} = \left(\sum_{i=1}^n \int_{i-1}^i |x_i|^p dt\right)^{\frac{1}{p}} = \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} = \|x\|_p$
- $\|g\|_q = \left(\int_0^n |g(t)|^q dt\right)^{\frac{1}{q}} = \left(\sum_{i=1}^n \int_{i-1}^i |y_i|^q dt\right)^{\frac{1}{q}} = \left(\sum_{i=1}^n |y_i|\right)^{\frac{1}{q}} = \|y\|_q$

$\therefore$ $\|fg\|_1 \leq \|f\|_p\|g\|_q$ implies $\sum_{i=1}^n |x_i y_i| \leq \|x\|_p\|y\|_q$. Therefore, Hölder's Inequality holds.

(b) Idea: Apply Minkowski's Inequality for $R[0,n]$ to some step functions.

Recall Minkowski's Inequality for $R[0,n]$: for any $f,g \in R[0,n]$, $\|f+g\|_p \le \|f\|_p + \|g\|_p$.

Given $x, y \in \mathbb{R}^n$, define f, g as in (a). Then

$(f+g)(t) = x_i + y_i$ for $t \in [i-1, i)$, $i = 1, \cdots, n$ and $(f+g)(n) = x_n + y_n$.

$\therefore \|f+g\|_p \le \|f\|_p + \|g\|_p$ implies $\|x+y\|_p \le \|x\|_p + \|y\|_p$. Therefore, Minkowski's Inequality holds.

Rmk

- (b) proves the triangle inequality axiom of $(\mathbb{R}^n, \|\cdot\|_p)$.
- From this proof, one can transport the equality cases conditions for $R[a,b]$ to the equality cases conditions for $\mathbb{R}^n$.
- Alternatively, one can prove Hölder's and Minkowski's Inequalities for $\mathbb{R}^n$ directly:

(a): Apply Young's Inequality in a similar way as in lecture note.

(b): Apply Hölder's Inequality for $\mathbb{R}^n$.

- Hölder's and Minkowski's Inequalities for $\mathbb{R}^n$ and $R[a,b]$ are special cases of Hölder's and Minkowski's Inequalities for "measure spaces" (which will be covered in MATH 4050: Real Analysis).

Q2) For each $p>0$, define the space of p-summable sequences l_p as

$l_p := \{(x_n)_{n=1}^{\infty} \mid x_n \in \mathbb{R} ; \sum_{n=1}^{\infty} |x_n|^p < +\infty\}$ and p-norm $\|\cdot\|_p : l_p \to \mathbb{R}$ as $\|x\|_p := \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}}$.

(a) Show that $(l_p, \|\cdot\|_p)$ is a normed space for $1 \leq p < +\infty$.

(b) Show that $(l_p, \|\cdot\|_p)$ is NOT a normed space for $0<p<1$.

Sol) (a) Idea: Apply Minkowski's Inequality to prove the triangle inequality axiom.

<u>Exercise</u> l_p is a real vector space under entrywise addition and scalar multiplication.

Checking $(l_p, \|\cdot\|_p)$ satisfy the axioms [N1]-[N3] for normed spaces:

[N1]: $\forall x \in l_p$, $\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} \geq 0$; $\|x\|_p = 0 \Leftrightarrow \forall n \in \mathbb{N}, |x_n| = 0 \Leftrightarrow x = 0$.

[N2]: $\forall x \in l_p, \forall \alpha \in \mathbb{R}$, $\|\alpha x\|_p = \left(\sum_{n=1}^{\infty} |\alpha x_n|^p\right)^{\frac{1}{p}} = \left(\sum_{n=1}^{\infty} |\alpha|^p |x_n|^p\right)^{\frac{1}{p}} = |\alpha| \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} = |\alpha| \|x\|_p$.

[N3]: $\forall x, y \in l_p$, $\forall N \in \mathbb{N}$, define $x^{(N)} = (x_1, \cdots, x_N)$, $y^{(N)} = (y_1, \cdots, y_N) \in \mathbb{R}^N$. Then

$$\left(\sum_{n=1}^{N} |x_n + y_n|^p\right)^{\frac{1}{p}} = \|x^{(N)} + y^{(N)}\|_p \leq \|x^{(N)}\|_p + \|y^{(N)}\|_p \quad \text{(by Q1b for } p>1 \text{; lecture note for } p=1\text{)}$$

$$= \left(\sum_{n=1}^{N} |x_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{N} |y_n|^p\right)^{\frac{1}{p}} \leq \|x\|_p + \|y\|_p$$

$\therefore$ Take $N \to +\infty$: $\|x+y\|_p \leq \|x\|_p + \|y\|_p$

$\therefore$ $(l_p, \|\cdot\|_p)$ is a normed space.

(b) Idea: Provide a counterexample to triangle inequality.

Showing [N3] is false: Choose $x = (1, 0, \cdots)$; $y = (0, 1, 0, \dots)$,

then $\|x\|_p = 1 = \|y\|_p$; $\|x+y\|_p = (1^p + 1^p)^{\frac{1}{p}} = 2^{\frac{1}{p}}$

$\because 0 < p < 1 \Rightarrow \|x+y\|_p = 2^{\frac{1}{p}} > 2 = \|x\|_p + \|y\|_p$.

$\therefore$ [N3] is false, hence $(\ell_p, \|\cdot\|_p)$ is NOT a normed space for $0 < p < 1$.

Rmk · (a) generalises the statement that $(\ell_1, \|\cdot\|_1)$ and $(\ell_2, \|\cdot\|_2)$ are normed spaces to arbitrary $p > 1$.

· In fact, one could take $p = \infty$ in the sense that $(\ell_\infty, \|\cdot\|_\infty)$ is the space of bounded sequences endowed with the sup-norm $\|\cdot\|_\infty$.

The fact that $(\ell_\infty, \|\cdot\|_\infty)$ is a normed space is shown in Tutorial 4, Remark 3.

· Exactly the same argument shows that for any $n \geq 2$,

(a) $(\mathbb{R}^n, \|\cdot\|_p)$ is a normed space for $p \geq 1$.

(b) $(\mathbb{R}^n, \|\cdot\|_p)$ is NOT a normed space for $0 < p < 1$.

Q3) Prove the Completion Theorem via Tutorial 4, Q2:

Given a metric space (X,d), there exists a completion (Y,Φ) of (X,d), where

- (Y,ρ) is a complete metric space.
- $\Phi:(X,d)\to(Y,\rho)$ is an isometric embedding such that $\overline{\Phi(X)}=Y$.

Sol) Idea: Apply the result of Tutorial 4, Q2.

Recall the result of Tutorial 4, Q2: there exists an isometric embedding

$\Phi:(X,d)\to(\mathcal{C}^b(X),d_\infty)$, where $(\mathcal{C}^b(X),d_\infty)$ is the space of bounded continuous functions on X endowed with the sup-metric d_∞.

Exercise Show that $(\mathcal{C}^b(X),d_\infty)$ is complete.

(Hint: similar proof as in showing $(\mathcal{C}[a,b],d_\infty)$ is complete as in the lecture.)

Define $(Y,\rho):=\left(\overline{\Phi(X)},\ d_\infty|_{\overline{\Phi(X)}}\right)$ as closed subspace of $(\mathcal{C}^b(X),d_\infty)$, so is complete.

Then $\Phi:(X,d)\to(Y,\rho)$ is an isometric embedding with $\overline{\Phi(X)}=Y$.

Rmk This proof of Completion Theorem is shorter than the lecture's one, but is also less explicit in the sense that $\overline{\Phi(X)}$ is not very well-understood.